MA1508E: Linear Algebra for Engineering

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Equivalent Statements

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent.

1. $\boldsymbol{A}$ is invertible
2. $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution
3. The RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.
5. $\det(\boldsymbol{A}) \neq 0$
6. The rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.
7. The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$
8. $rank(\boldsymbol{A}) = n$
9. 0 is not an eigenvalue of $\boldsymbol{A}$
10. $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution for each $\boldsymbol{b} \in \mathbb{R}^n$
11. Nullspace of $\boldsymbol{A} = \{0\}$
12. $\text{nullity}(\boldsymbol{A}) = 0$
13. $\boldsymbol{A}^T$ is invertible.
14. The REF of $\boldsymbol{A}$ has n leading entries.
15. The columns/rows of $\boldsymbol{A}$ are linearly independent.
16. The columns/rows of $\boldsymbol{A}$ span $\mathbb{R}^n$.
17. The column space/row space of $\boldsymbol{A}$ is $\mathbb{R}^n$.
18. The dimension of the column space/row space is n.
19. The only vector orthogonal to the col/row space of $\boldsymbol{A}$ is $\boldsymbol{0}$.

Week 1: Introduction to Linear Systems

Elementary Row Operations

1. Multiplying the ith row by a non-zero constant c. (cR_i)
2. Interchanging the ith row and the jth row. ($R_i \leftrightarrow R_j$)
3. Adding k times the ith row to the jth row. ($R_j + kR_i$)

Row Echelon Form

1. For two consecutive non-zero rows, the first nonzero entry of the lower row is further right than the one on top.
2. Pivot column is the column that consist of a pivot point.

Week 2: Linear Systems and Gauss Jordan Elimination

Types of Linear Systems

Inconsistent – Last column is a pivot column.

Unique Solution – Last column is not pivot and every column on the left is pivot.

Infinitely Many Solutions – Last column is not pivot and some column on the left is not pivot

Gaussian Elimination

1. Locate the leftmost column that does not consist of entirely zeroes.
2. Interchange the top row with another row to bring a non-zero entry to the top of the column.
3. For each row below the top row, add a suitable multiple of the top row so that the entries below the leading entry of the top row becomes zero.
4. Cover the top row matrix and begin with (1.) applied to the submatrix that remains. Continue till matrix is in row echelon form.

Jordan Elimination

5. Multiply a suitable constant to each row so that all the leading entries become 1.
6. Begin with the last non-zero entry and working upwards, add suitable multiples of each row to the row above to introduce zeroes above the leading entry.

Week 3: Matrix Operations

1. Matrices A and B are considered equal when $a_{ij} = b_{ij}\ \forall\ i, j$
2. For matrix A and B that are $m \times n$,
   1. $A + B = (a_{ij} + b_{ij})$
   2. $A - B = (a_{ij} - B_{ij})$
   3. $cA = (ca_{ij})$
   4. $-A = (-1)A$
3. Matrices of different sizes cannot be added or subtracted.

A, B and C are matrices, and c and d are scalar values.

Commutative	-	$A + B = B + A$
Associative	-	$A + (B + C) = (A + B) + C$
	-	$c(dA) = (cd)A = d(cA)$
	-	$A(BC) = (AB)C$
	-	$c(AB) = (cA)B = A(cB)$
Distributive	-	$c(A + B) = cA + cB$
	-	$(c + d)A = cA + dA$
	-	$A(B_1 + B_2) = AB_1 + AB_2$
	-	$(C_1 + C_2)A = C_1A + C_2A$

Matrix Multiplication

Let $A = (a_{ij})_{m \times p}$ and $B = (b_{ij})_{p \times n}$.

AB will be an $m \times n$ matrix, whose (i, j)-entry is given by

$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$

Pre-multiplication A to B $\rightarrow AB$

Post-multiplication A to B $\rightarrow BA$

Matrix Inverses and Inverse Laws

A is said to be invertible if there exists another square matrix B of the same order such that $AB = BA = I_n$. If such an B exists, it is called an inverse of A. Else, A is said to be singular.

Let A be an invertible square matrix.

1. $A^{-1}AB = B$
2. $AA^{-1}B = B$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $(A^{-1})^{-1} = A$
6. $(AB)^{-1} = B^{-1}A^{-1}$
7. $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$
8. $(A^n)^{-1} = A^{-n}$

Matrix Transposition and Rules

Let $A = (a_{ij})_{m \times n}$. The transpose of A, denoted as A^T is an $n \times m$ matrix whose (i, j)-entry is a_{ji}.

A matrix is symmetrical if $A = A^T$.

1. $(A^T)^T = A$
2. $(aA)^T = aA^T$
3. $(A + B)^T = A^T + B^T$
4. $(AB)^T = B^TA^T$

Week 4: Elementary Matrices

For any matrix A, there exists elementary matrices $E_kE_{k-1} \dots E_1A$ which forms the RREF of A.

A and B are row equivalent $\Leftrightarrow$ they have similar REF (or same unique RREF)

To obtain elementary matrices, perform ERO on identity matrices.

Definition of Elementary Matrices

A square matrix is an elementary matrix if it can be obtained from I by performing 1 ERO. All elementary matrices are invertible, and its inverse is also an elementary matrix.

Computing Inverses using Gaussian Elimination

If a matrix A is invertible, then $E_kE_{k-1} \dots E_1A = I_n$. Since $E_kE_{k-1} \dots E_1$ and A are square matrices of same sizes, $E_kE_{k-1} \dots E_1 = A^{-1}$. Hence, to find out what the elementary matrices are, perform the same elementary matrix multiplications on an I simultaneously with A until A forms I. If A is singular, then its RREF will not be I, and hence there will not be an inverse.

Week 5: Determinants and Cofactor Expansion

Definition of Determinants and Cofactor Expansion

$$\det(A) \neq 0 \Leftrightarrow \text{A is invertible.}$$

Let A be a square matrix of order n, and M_{ij} be a square matrix of order $n - 1$ obtained by removing the ith row and the jth column of A.

$$\det(A) = \begin{cases} a_{11} & if\ n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & if\ n \geq 2 \end{cases}$$

where $A_{ij} = (-1)^{i+j}\det(M_{ij})$.

A_{ij} is defined as the (i, j)-cofactor of A.

Cofactor matrix is the respective cofactor for each entry in A.

Determinants of Special Matrices

Triangular Matrices - $\det(A)$ is the product of the diagonal entries.

Square Matrix of 2 Identical Rows/Columns - $\det(A) = 0$

Determinants and ERO

$\boldsymbol{E_1 = kR_i}$	$\boldsymbol{E_2 = R_i \leftrightarrow R_j}$	$\boldsymbol{E_3 = R_i + kR_j}$
$\det(E_1) = k$	$\det(E_2) = -1$	$\det(E_3) = 1$

Properties of Determinants

Let A and B be two square matrices of order n and c is a scalar.

1. $\det(cA) = c^n\det(A)$
2. $\det(AB) = \det(A)\det(B)$
3. $\det(A^{-1}) = \frac{1}{\det(A)}$
4. $\det(A) = \det(A^T)$

Adjoint Method

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{\det(A)}(\text{cofactor matrix of } A)^T$$

Cramer's Rule

$Ax = b$ is a linear system where A is an invertible square matrix of order n. For $i = 1, 2, \dots, n$, A_i is a square matrix of order n where the ith column is replaced by b.

The unique solution of $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where $x_i = \frac{\det(A_i)}{\det(A)}, i = 1, 2, \dots, n$

Week 6: Vectors and Span

Properties of Euclidean Vectors

1. $v = (v_1, v_2, \dots, v_n), v_i \in \mathbb{R}$
2. $v \in \mathbb{R}^n$ is called n-vector
3. $u = (u_1\ \ u_2\ \ \cdots\ \ u_n)$
4. $u = v \Leftrightarrow (u_1\ \ \cdots\ \ u_n) = (v_1\ \ \cdots\ \ v_n)$
5. Length: $|v| = \sqrt{v_1^2 + \cdots + v_n^2}$
6. Distance: $|u - v| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$
7. Angle: $\theta = \cos^{-1}\frac{u \cdot v}{|u||v|}$
8. Dot Product:
   a. $u \cdot v = u_1 \cdot v_1 + \cdots + u_n \cdot v_n$
   b. $u \cdot v = uv^T = (u_1\ \ \cdots\ \ u_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

Orthogonal Vectors, Orthogonal Sets, and Orthonormal Sets

- Two vectors are orthogonal if $u \cdot v = 0$.
- Set $S = \{v_1, \dots, v_k\}$ is orthogonal if every pair of distinct vectors are orthogonal.
- Set S is orthonormal if it is orthogonal, and every vector in S is a unit vector.

Linear Combination

w is a linear combination of v and u if there exists c_1, c_2 such that $w = c_1v + c_2u$.

Span

Span is a set of all possible linear combinations of vectors in the set. e.g.

$$S = \{u_1, \dots, u_k \mid u_i \in \mathbb{R}^n\}$$

$$\text{span}(S) = \{c_1u_1 + c_2u_2 + \cdots + c_ku_k \mid c \in \mathbb{R}\}$$

To check if $\text{span}(S) \neq \mathbb{R}^n$, check whether the augmented matrix formed by S and any $x \in \mathbb{R}^n$ is inconsistent. (i.e. last col. Is pivot)

Redundant Vector Theorem

If u_k is a linear combination of $u_1, \dots, u_{k-1}$,

then $\text{span}\{u_1, \dots, u_k\} = \text{span}\{u_1, \dots, u_{k-1}\}$

Week 7: Subspaces

$V \subseteq \mathbb{R}^n$ subspace of $\mathbb{R}^n \Leftrightarrow$ some $S = \{u_1, \dots u_k\}, u_i \in \mathbb{R}^n$, $\text{span}(S) = V \Rightarrow \{c_1u_1 + \cdots + c_ku_k \mid c_i \in \mathbb{R}\}$

Properties

1. V is the subspace spanned by $S = \{u_1, \dots, u_k\}$.
2. V must contain the zero vector.
3. Subspace must be expressed as a linear span

NOTE: Vectors in S need not be linearly independent.

Zero Space

$\{0\} = \text{span}\{0\}$ is a subspace of $\mathbb{R}^n$, a.k.a. zero space of $\mathbb{R}^n$.

The only subspace of $\mathbb{R}^n$ with finite number and thus least vectors.

Linear Independence

For $S \subseteq \mathbb{R}^n$, if $c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$ only has the trivial solution ($c_i = 0\ \forall i \leq k$), then S is a set of *linearly independent* vectors. Else, vectors in S are *linearly dependent*.

Checking for Linear Independence

1. Form vector equation → Form linear system
2. Solve using GJE
   a. Trivial Solution – Vectors are linearly independent
   b. Non-trivial Solutions – Vectors are linearly dependent

Guaranteed Linear Dependence

1. If $u \in S$ is a redundant vector, then S is linearly dependent.
2. If $|S| > n$ in $\mathbb{R}^n$, S is definitely linearly dependent.

The solution set for a homogenous system of linear equations with n unknowns is a subspace of $\mathbb{R}^n$.

Week 8: Vector Spaces, Bases, Dimensions

Vector Spaces/Bases

V is a vector space if $V \subseteq \mathbb{R}^n$. If S is a basis for V:

1. $\text{span}(S) = V$
2. S is linearly independent. (check if only trivial solution)

- If $S = \{u_1, \dots, u_k\}$ is a basis for V, any $v \in V$ can be written as $v = c_1u_1 + \cdots + c_ku_k$ in a unique $c_1, \dots, c_k$ that make up a coordinate vector for V.
- If $S_1 \neq S_2$ are basis for V, then $v \in S_1 \neq v \in S_2$
- Any set cannot span $V \in \mathbb{R}^k$ if it has less than k vectors

Dimension

Dimension of $V = \dim(V) = |V| =$ number of vectors in a basis

S is a basis for $V \Leftrightarrow$

1. S is linearly independent and $|S| = k$ where $|V| = k$
2. $\text{span}(S) = V$

- Bases of the same vector space have same number of vectors.
- If X is a subspace of Y, then the dimension of X is no larger than the dimension of Y. (i.e. $dim(X) \leq \dim(Y)$)
- Rows of A are basis for $\mathbb{R}^n \Leftrightarrow$ columns of A are basis for $\mathbb{R}^n$ (means A is invertible)

Week 9: Row Space, Column Space, Rank, Nullspace

Row Space and Basis for Row Spaces

Let A and B be row equivalent matrices. (i.e. row space of A and B are identical)

Performing ERO on A does not change the row space.

i.e. $A \xrightarrow{cR_i} B$ or $A \xrightarrow{R_i \leftrightarrow R_j} B$ or $A \xrightarrow{R_i + cR_j} B$ does not affect the spans, hence have equivalent row spaces.

To find the basis of A:

1. Find the row echelon form R of A by GE. The non-zero rows of R will be a basis for the row space of A.

NOTE: For rows that form zero-rows, it is a linear combination of the non-zero rows.

Column Space and Basis for Column Spaces

NOTE: ERO preserves the row space but ***NOT*** *the column space.*

To find the column space of A,

Method 1: Perform ERO on the transpose of the matrix. (i.e. col. space of A = row space of A^T)

Method 2: The columns of A that correspond to the pivot columns in r.e.f. R will be the basis for the col. space. e.g.

Basis for col space:

$$\begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1.5 & -3 & 1.5 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Rank of Matrix

The row space and col. space of a matrix have same dimensions.

- Basis of the row space are pivot columns, and basis of column space are the corresponding pivot columns, hence both will have the same dimension.

Properties:

1. $\text{rank}(A)$ is the dimension of the row/column space
2. For an $m \times n$ matrix A, $\text{rank}(A) \leq \min\{m, n\}$.
3. If $\text{rank}(A) = \min\{m, n\}$, then A is full rank.
4. $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$
5. $\text{rank}(A^T) = \text{rank}(A)$ 6. $\text{rank}(AB)^T = \text{rank}(B^T A^T)$

Linear Systems and Column Spaces

Given a linear system $Ax = b$,

- $Ax = b$ is consistent → There is a solution $x = \{x, y, z\}$ to satisfy the linear system.
- b is a linear combination of the columns of A (i.e. b belongs in the col. space of A)

Hence,

1. Linear system will be consistent if b lies in col. space of A.
2. $Ax = b$ is consistent $\Leftrightarrow \text{rank}(A) = \text{rank}(A \mid b)$

Nullspace of a Matrix

Let A be an $m \times n$ matrix. $Ax = 0$ is a homogenous linear system with n variables (i.e. there are n col. vectors)

- The solution set/solution space/nullspace of $Ax = 0$ is a subspace of $\mathbb{R}^n$
- The dimension of the nullspace of A, $\text{nullity}(A) \leq n$ since it is a subspace of $\mathbb{R}^n$.

Dimension Theorem for Matrices

Pivot Columns	Non-pivot Columns	
No. of pivot columns	No. of non-pivot columns	
No. of leading entries in R	No. of arbitrary parameters in a general solution to $Ax = 0$	
$\text{rank}(A)$	$\text{nullity}(A)$	$= n$

General Solutions of Linear Systems

The entire solution set for $Ax = b$ can be obtained by adding:

1. The nullspace of A (i.e. solution set/space for $Ax = 0$)
2. A particular solution to $Ax = b$

If $\text{nullity}(A) = 0$, then it has only 1 solution of x for $Ax = b$.

However, if $\text{nullity}(A) > 0$, then it has infinitely many solutions.

$$Solution\ Set = Particular\ Solution + General\ Solution$$

Week 10: Orthogonal Bases, Projection, Least Squares Solution

Orthogonal and Orthonormal Bases

A basis S is an orthogonal/orthonormal basis if S is an orthogonal/orthonormal set.

Determining Orthogonal Bases

For vector space V with $dim(V) = k$, if S is an orthogonal set of non-zero vectors in V, and $|S| = k$, then S is an orthogonal basis for V.

Vectors as a Linear Combination of Orthogonal/Orthonormal Basis

If $S = \{u_1, \ldots, u_k\}$ is an orthogonal basis for V then any $w \in V$ can be represented by:

$$w = \left(\frac{w \cdot u_1}{||u_1||^2}\right)\boldsymbol{u_1} + \cdots + \left(\frac{w \cdot u_k}{||u_k||^2}\right)\boldsymbol{u_k}$$

For orthonormal basis, $||u_i||^2 = 1 \ \forall \ i \leq k$ as vectors in S are unit vectors.

Vectors Orthogonal to a Space

- u is orthogonal to $V \Leftrightarrow u$ is orthogonal to all vectors in V.
- Finding an orthogonal vector is like finding a normal to the plane in $\mathbb{R}^3$ space. $V = \{(x, y, z) \mid ax + by + cz = 0\}$
  Let $n = (a, b, c)$. Then $V = \{u \in \mathbb{R}^3 \mid u \cdot n = 0\}$ where n is the orthogonal (normal) vector to V.
- Scalar multiples of n are also orthogonal vectors to V.

Orthogonal Projection Theorem

Let V be a subspace of $\mathbb{R}^n$. Every $u \in \mathbb{R}^n$ can be uniquely written as $u = n + p$, where n is orthogonal to V and p is the projection of u onto V. (if $u \in V, n = 0$)

To find p, simply find the linear combination of u onto V. Hence to find the orthogonal vector n to V, $n = u - p$.

Gram-Schmidt Process (Converting Bases to Orthogonal Bases)

Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for vector space V.

Let $v_1 = u_1$;

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{||v_1||^2}v_1; \qquad v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{||v_1||^2}v_1 + \frac{u_3 \cdot v_2}{||v_2||^2}v_2\right);$$

$\vdots$

$$v_k = u_k - \left(\frac{u_k \cdot v_1}{||v_1||^2}v_1 + \frac{u_k \cdot v_2}{||v_2||^2}v_2 + \cdots + \frac{u_k \cdot v_{k-1}}{||v_{k-1}||^2}v_{k-1}\right);$$

Best Approximation

p is the best approximation of $u \in V$ if $d(u, p) \leq d(u, v) \ \forall \ v \in V$, where p is the projection of u onto V.

Least Squares Solution

Let $Ax = b$ be a linear system where A is an $m \times n$ matrix.

- A vector $u \in \mathbb{R}^n$ is the least squares solution to the linear system if $||b - Au|| \leq ||b - Av|| \ \forall \ v \in \mathbb{R}^n$
- $Ax = b$ will only be consistent if b belongs to the col. space of A. (least squares solution will be the exact solution)

Least Squares Solution Methods

Let $Ax = b$ be a linear system.

x is the least squares solution to $Ax = b \Leftrightarrow$

x is a solution to $A^T Ax = A^T b$ (a.k.a. the normal equation)

Week 11: Eigenvalues/Eigenvectors/Eigenspaces, Diagonalization

Eigenvalues are used to diagonalize square matrices, which can then be used to find high power of matrices easily.

Definition

A non-zero column vector $u \in \mathbb{R}^n$ is an eigenvector of A if

$$Au = \lambda u \quad \text{for some scalar } \lambda.$$

Eigenvalue λ is a scalar value that makes $(\lambda I - A)$ singular.

All scalar multiples of u will also be an eigenvector of A associated with the same eigenvalue.

Finding Eigenvalues

NEVER PERFORM ERO TO CONVERT TO TRIANGULAR MATRIX

Since u is a non-zero column vector, $det(\lambda I - A) = 0$.

To find all eigenvalues, solve the characteristic equation of A.

For triangular matrices, eigenvalues of A are its diagonal entries.

Eigenspaces

The eigenspace of A, E_λ, associated with λ is the solution space of $\lambda I - A = 0$.

Diagonalizable Matrices

A is diagonalizable if:

1. There exists invertible matrix P such that $P^{-1}AP = D$
2. A has n linearly independent eigenvectors
3. A has $k \leq n$ distinct eigenvalues.

Method to determine whether matrix is diagonalizable:

1. Solve $\det(\lambda I - A) = 0$ to find all k distinct λ of A.
2. For each λ_i, find a basis S_{λ_i} for eigenspace E_{λ_i}.
3. Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup \ldots \cup S_{\lambda_k} = \{u_1, u_2, \ldots, u_n\}$
   a. $|S| < n$, then A is not diagonalizable
   b. $|S| = n$, then $P = (u_1 \quad u_2 \quad \cdots \quad u_n)$ diagonalizes A.
   c. $|S| > n$, error in calculation

Week 12: System of Linear D.E., Complex Vectors

Setting up a Linear Recurrence

Set up a transition matrix A, i.e.

$$\begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \end{pmatrix}$$

1. Determine values of A by relating a_n to the recurrence relation (e.g. $a_n = a_{n-1} + a_{n-2}$)
2. Calculate eigenvalues of A, then form equations of $a_n = A\lambda_0^n + B\lambda_1^n + \cdots$
3. Solve using initial values of a_n and all λ for $A, B, \ldots$.

Fundamental Set

For square matrix A of order n,

1. There exist a fundamental set of solutions to $Y' = AY$.
2. n linearly independent functions in fundamental set S.
3. Each solution in S is a unique linear combination of n functions in S.
4. S is an n-dimensional vector space of functions.
5. If vector Y_0 is specified, then the initial value problem is to construct a unique Y (in S) such that $Y' = AY$ and $Y(0) = Y_0$

Initial Value Problem

1. Form D.E. $y_1'(t)$ to $y_i'(t)$.
2. Find all eigenvalues of $A, \lambda_1, \ldots, \lambda_i$.
3. Find all linearly independent eigenvectors associated with λ_i
4. Construct linear combinations of solution X_1 to X_i.
5. General Solution: $Y = k_1 e^{\lambda_1 t} x_1 + k_2 e^{\lambda_2 t} x_2 + \cdots + k_i e^{\lambda_i t} x_i$
6. Use given initial conditions to solve for $k_1, \ldots, k_i$.

Properties of Complex Vectors

Let u, v, w, z be vectors in $\mathbb{C}^n$ and if k is a scalar, then

1. $\overline{ku} = \bar{k}\bar{u}$ 2. $\overline{u + v} = \bar{u} + \bar{v}$
3. $\overline{u - v} = \bar{u} - \bar{v}$ 4. $u \cdot v = \overline{v \cdot u}$
5. $u \cdot (v + w) = u \cdot v + u \cdot w$ 6. $k(u \cdot v) = (ku) \cdot v$
7. $u \cdot v = u_1\overline{v_1} + \cdots + u_n\overline{v_n}$ 8. $u \cdot kv = \bar{k}(u \cdot v)$
9. $||v|| = \sqrt{v \cdot v} = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n^2|}$
10. $v \cdot v \geq 0$ and $v \cdot v = 0 \Leftrightarrow v = 0$
11. $|z| = \sqrt{Re(z)^2 + Im(z)^2}$ 12. $z\bar{z} = |z|^2$
13. $Re(z) = |z| \cos\theta$ 14. $Im(z) = |z| \sin\theta$
15. $z = |z|(\cos\theta + i \sin\theta)$

Complex Eigenvalues

The conjugate of a complex eigenvalue is also an eigenvalue of A, and the conjugate eigenvector is also the corresponding vector.

$$e^{\lambda t}x = e^{(a+ib)t}x = e^{at}(\cos bt + i \sin bt)(\text{Re}(x) + i\ \text{Im}(x))$$
$$\text{Re}(e^{\lambda t}x) = e^{at}((\cos bt)\text{Re}(x) - (\sin bt)\text{Im}(x)) = Y_1$$
$$\text{Im}(e^{\lambda t}x) = e^{at}((\cos bt)\text{Im}(x) + (\sin bt)\text{Re}(x)) = Y_2$$

Types of Matrices and Factorisation Techniques

Types of Matrices

Idempotent	**Orthogonal**	**Symmetric**
$A^2 = A$	$A^{-1} = A^T$	$A = A^T$

Vector Inequalities

Cauchy-Schwarz Inequality – $|u \cdot v| \leq ||u||||v||$

Triangle Inequality – $||u + v|| \leq ||u|| + ||v||$

LU Factorisation

Linear systems involving triangular coefficient matrices are easy to deal with. LU factorization can decompose matrices to upper and lower triangular matrices to perform substitutions easily.

To convert matrix $A \rightarrow LU$:

$$L = E_k^{-1}E_{k-1}^{-1} \ldots E_1^{-1}I, \quad A \xrightarrow{GE} U$$
$$Ax = b \Rightarrow L(Ux) = b$$

Solving for $y = Ux$ and $Ly = b$, one can find b easily. This will be useful for multiple values of b.

Note: Not all matrices can be written in LU form.

QR Factorisation

QR Factorisation can be useful in finding least squares solution to a linear system $Ax = b$. Q is a matrix with orthonormal columns and R is an upper triangular matrix with positive diagonal entries.

$$A^T Ax = A^T b \Rightarrow (QR)^T(QR)x = (QR)^T b \Rightarrow x = R^{-1}Q^T b$$

This implies that to find a least squares solution, if we have $A = QR$, then $x' = R^{-1}Q^T b$.

To convert $A \rightarrow QR$:

1. Use Gram-Schmidt Process to transform the column space of matrix A into an orthonormal column space, forming into a matrix Q.
2. Express each column of A as a linear combination of columns in Q. Form upper triangular matrix R with rows being each linear combination.