First Order ODE

1. Reduction to Separable Form

$$y' = g\left(\frac{y}{x}\right)$$

Set $\frac{y}{x}$ to be v, then $y = vx$ and $y' = v + xv'$

2. Linear Change of Variable

$$y' = f(ax + by + c)$$

where f is continuous and $b \neq 0$, solve by setting

$$u = ax + by + c$$

Linear First Order ODE

1. Integrating Factor

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Obtain the integrating factor R,

$$R(x) = e^{\int P(x)\,dx}$$

Solve by setting:

$$Ry' + RPy = RQ \text{ or } (Ry)' = RQ$$

2. Reduction to Linear Form (Bernoulli Equation)

$$y' + P(x)y = Q(x)y^n$$

Solve by rewriting:

$$y^{-n}y' + y^{1-n}P(x) = q(x)$$

Set $y^{1-n} = z$, so $z' = (1-n)y^{-n}y'$,

$$z' + (1-n)P(x)z = (1-n)q(x)$$

Linear Second Order Homogenous ODE

1. Characteristic Equation

$y'' + ay' + by = 0$ has the characteristic equation:

$$\lambda^2 + a\lambda + b = 0$$

$$\lambda_1\lambda_2 = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$$

2. Distinct Roots

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

3. Real Double Roots

$$y = (A + Bx)e^{-\frac{ax}{2}}$$

4. Complex Roots ($\lambda_1\lambda_2 = \frac{a}{2} \pm i\omega$)

$$y = (A\cos\omega x + B\sin\omega x)e^{-\frac{ax}{2}}$$

Linear Second Order Nonhomogeneous ODE

1. General Solution

$$y'' + P(x)y' + Q(x)y = R(x)$$

$$y(x) = y_h(x) + y_p(x)$$

2. Method of Undetermined Coefficient (Polynomial)

$$y'' + ay' + y = x^2 + x + 2$$

Try $y = Ax^2 + Bx + C$:

$$Ax^2 + (B - 8A)x + 2A - 4B + C = x^2 + x + 2$$

Compare coefficients:

$$A = 1, B = 9, C = 36$$

$$\therefore y_p(x) = x^2 + 9x + 36$$

3. Method of Undetermined Coefficient (Exponential)

$$y'' - 4y' + 2y = 2x^3e^{2x}$$

Try $y = ue^{2x}$

$$u''e^{2x} - 2ue^{2x} = 2x^3e^{2x}$$

$$u'' - 2u = 2x^3$$

Solve using Undetermined Coefficient (Polynomial), $u = -x^3 - 3x$

$$\therefore y_p(x) = (-x^3 - 3x)e^{2x}$$

4. Method of Undetermined Coefficient (Trigonometry)

$$y'' + 4y = 16x\sin 2x$$

Solve instead $z'' + 4z = 16xe^{i2x}$ using Method of Undetermined Coefficient (Exponential):

$$u'' + 4iu' = 16x$$

Solve using Undetermined Coefficient (Polynomial), $u = -2ix^2 + x$ and $z = (-2ix^2 + x)e^{i2x}$

$$\therefore y_p(x) = Imz = x\sin 2x - 2x^2\cos 2x$$

5. Method of Variations of Parameters

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

$$y_h(x) = Ay_1(x) + By_2(x)$$

$$u = -\int \frac{y_2 r}{y_1y_2' - y_1'y_2}\,dx$$

$$v = \int \frac{y_1 r}{y_1y_2' - y_1'y_2}\,dx$$

Simple Harmonic Motion

$$mL\ddot{\theta} = -mg\sin\theta$$

$$\frac{2\pi}{\omega} = 2\pi\sqrt{L/g}$$

Unstable Equilibrium ($\theta = \pi$):

$$\sin\theta \approx -(\theta - \pi)$$

$$mL\ddot{\theta} = mg(\theta - \pi)$$

$$\theta - \pi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t}$$

Stable Equilibrium ($\theta = 0$):

$$\sin\theta \approx \theta$$

$$mL\ddot{\theta} = -mg\theta$$

$$\ddot{\theta} = -\frac{g}{L}\theta = -\omega^2\theta$$

$$\theta = A\cos\omega t + B\sin\omega t \text{ or } \theta = A\cos(\omega t - \delta)$$

Damping:

For a SHM with equation:

$$A\ddot{x} + B\dot{x} + Cx = 0$$

Underdamped if $B^2 - 4AC < 0$

Critically damped if $B^2 - 4AC = 0$

Overdamped if $B^2 - 4AC > 0$

Forced Oscillations

No force applied:

$$m\ddot{x} = -kx = -m\omega^2 x$$

Force applied:

$$m\ddot{x} + kx = F_0\cos\alpha t$$

$$x = \frac{2F_0/m}{\alpha^2 - \omega^2}\sin\left[\left(\frac{\alpha - \omega}{2}\right)t\right]\sin\left[\left(\frac{\alpha + \omega}{2}\right)t\right]$$

At resonance ($\alpha \to \omega$):

$$x = \frac{F_0 t}{2m\omega}\sin(\omega t)$$

Conservation

$$m\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = -kx$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Malthus Model

$$\frac{dN}{dt} = (B - D)N = kN$$

$$N(t) = N_0e^{kt}$$

Logistic Model

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

$$N_\infty = \frac{B}{s}$$

$$N(t) = \frac{B}{s + \left(\frac{B}{N_0} - s\right)e^{-Bt}} \leftrightarrow N(t) < N_\infty$$

$$N(t) = \frac{B}{s - \left(s - \frac{B}{N_0}\right)e^{-Bt}} \leftrightarrow N(t) > N_\infty$$

Logistic Model with Harvesting

$$\frac{dN}{dt} = BN - DN - E = N(B - sN) - E$$

Population wiped out:

$$E > \frac{B^2}{4s}$$

Population at stable equilibrium $\beta_2 \leftrightarrow \beta_2 > \beta_1$:

$$E < \frac{B^2}{4s}$$

$$\beta_1\beta_2 = \frac{-B \pm \sqrt{B^2 - 4Es}}{-2s}$$

Population at unstable equilibrium:

$$E = \frac{B^2}{4s}$$

Wave Equation

$$c^2\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial t^2}$$
$$y(t,x)=\frac{1}{2}[f(x+ct)+f(x-ct)]$$

Heat Equation

$$u_t=c^2u_{xx}$$
$$u(0,t)=u(L,t)=0$$
$$u(x,0)=f(x)$$

Laplace Transformation

$\mathcal{L}(1)=\frac{1}{s}$	$\mathcal{L}(e^{at})=\frac{1}{s-a}$
$\mathcal{L}(t^n)=\frac{n!}{s^{n+1}}$	$\mathcal{L}(\sqrt{t})=\frac{\sqrt{\pi}}{2s^{3/2}}$
$\mathcal{L}(\cosh at)=\frac{s}{s^2-a^2}$	$\mathcal{L}(\sinh at)=\frac{a}{s^2-a^2}$
$\mathcal{L}(\cos at)=\frac{s}{s^2+a^2}$	$\mathcal{L}(\sin at)=\frac{a}{s^2+a^2}$
$\mathcal{L}(t\cos at)=\frac{s^2-a^2}{(s^2+a^2)^2}$	
$\mathcal{L}(t\sin at)=\frac{2as}{(s^2+a^2)^2}$	
$\mathcal{L}(\cos at-at\sin at)=\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	
$\mathcal{L}(\cos at+at\sin at)=\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$	
$\mathcal{L}(\sin at-at\cos at)=\frac{2a^3}{(s^2+a^2)^2}$	
$\mathcal{L}(\sin at+at\cos at)=\frac{2a^2}{(s^2+a^2)^2}$	
$\mathcal{L}(\sin(at+b))=\frac{s\sin b+a\cos b}{s^2+a^2}$	
$\mathcal{L}(\cos(at+b))=\frac{s\cos b-a\sin b}{s^2+a^2}$	
$\mathcal{L}(e^{at}t^n)=\frac{n!}{(s-a)^{n+1}}$	
$\mathcal{L}(e^{at}\cos\omega t)=\frac{s-a}{(s-a)^2+\omega^2}$	
$\mathcal{L}(e^{at}\sin\omega t)=\frac{\omega}{(s-a)^2+\omega^2}$	
$\mathcal{L}(e^{at}\cosh\omega t)=\frac{s-a}{(s-a)^2-\omega^2}$	
$\mathcal{L}(e^{at}\sinh\omega t)=\frac{\omega}{(s-a)^2-\omega^2}$	
$\mathcal{L}(f(t-a)u(t-a))=e^{-as}\mathcal{L}(f(t))$ $\therefore\ \mathcal{L}(u(t-a))=\frac{e^{-as}}{s}$	
$\mathcal{L}(\delta(t-a))=e^{-as}$ $\therefore\ \mathcal{L}^{-1}(1)=\delta(t)$	

Laplace Transformation of Derivatives

$\mathcal{L}(f')$	$s\mathcal{L}(f)-f(0)$
$\mathcal{L}(f'')$	$s^2\mathcal{L}(f)-sf(0)-f'(0)$
$\mathcal{L}(f''')$	$s^3\mathcal{L}(f)-s^2f(0)-sf'(0)-f''(0)$

Laplace Transformation of Integrals

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)=\frac{1}{s}\mathcal{L}(f)$$

Trigonometric Identities

$\csc x=\frac{1}{\sin x}$	$\sec x=\frac{1}{\cos x}$
$\tan x=\frac{\sin x}{\cos x}$	$\cot x=\frac{\cos x}{\sin x}$
$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$	
$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$	
$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$	
$\sin 2A=2\sin A\cos A$	
$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$	
$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$	
$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$	
$\sin A-\sin B=2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$	
$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$	
$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$	
$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$	
$\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$	
$\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$	
$\cos A\sin B=\frac{1}{2}[\sin(A+B)-\sin(A-B)]$	

Hyperbolic Functions

$\cosh x=\frac{e^x+e^{-x}}{2}$	$\sinh x=\frac{e^x-e^{-x}}{2}$
$\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$	$\cosh^{-1}x=\ln\left(x\pm\sqrt{x^2-1}\right)$
$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{e^{2x}-1}{e^{2x}+1}$	
$\tanh^{-1}x=\frac{1}{2}\ln\frac{1+x}{1-x}$	

Partial Differential Equation

$$u_y=2u_{xx}, u(0,t)=u(3,t)=0, u(x,0)=5\sin 4\pi x$$

1. Set $u(x,y)=X(x)Y(y)$, $u_{xx}=X''(x)Y(y)$ and $u_y=X(x)Y'(y)$
2. Substitute u, u_{xx} and u_y into equation to obtain
$X''-kX=0$ and $Y'-2kY=0$
3. Solve to obtain $X(x)=a\cos\sqrt{-k}x+b\sin\sqrt{-k}x$ and $Y(y)=Ae^{2ky}$
4. Using boundary conditions, $X(0)=a=0, X(3)=b\sin 3\sqrt{-k}=0$
5. $\sqrt{-k}=\frac{n\pi}{3}, k=\frac{-n^2\pi^2}{9}, \therefore u_n(x,y)=b_ne^{-\frac{2n^2\pi^2 y}{9}}\sin\frac{n\pi x}{3}$
6. $\because\ u(x,0)=5\sin 4\pi x, n=12$
7. $\therefore u_p(x,y)=5e^{-32\pi^2 y}\sin 4\pi x$

Integration and Differentiation Techniques

$f(x)$	$f'(x)$
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$
$\csc x$	$\csc x\cot x$
$\sec x$	$\sec x\tan x$
$\sinh^{-1}x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1}x$	$\pm\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1}x$	$\frac{1}{1-x^2}$
$f(x)g(x)$	$f(x)g'(x)+f'(x)g(x)$
$\frac{f(x)}{g(x)}$	$\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$

$f(x)$	$\int f(x)\,dx$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\tan x$	$\ln(\sin x)$
$\csc x$	$-\ln(\csc x+\cot x)$
$\sec x$	$\ln(\sec x+\tan x)$
$\sin^2 ax$	$\frac{x}{2}-\frac{\sin 2ax}{4x}$
$\cos^2 ax$	$\frac{x}{2}+\frac{\sin 2ax}{4x}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{x+a}{x-a}$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\frac{x-a}{x+a}$
$uv'\,dx$	$uv-\int u'vdx$

Partial Fractions

$\frac{px+q}{(ax+b)(cx+d)}=\frac{A}{ax+b}+\frac{B}{cx+d}$
$\frac{px^2+qx+r}{(ax+b)(cx+d)^2}=\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}$
$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)}=\frac{A}{ax+b}+\frac{Bx+C}{x^2+c^2}$